

# Interfacial fluctuations near the critical filling transition

A.Bednorz and M.Napiórkowski  
Instytut Fizyki Teoretycznej, Uniwersytet Warszawski,  
00-681 Warszawa, Hoża 69, Poland

February 1, 2008

## Abstract

We propose a method to describe the short-distance behavior of an interface fluctuating in the presence of the wedge-shaped substrate near the critical filling transition. Two different length scales determined by the average height of the interface at the wedge center can be identified. On one length scale the one-dimensional approximation of Parry et al. [10] which allows to find the interfacial critical exponents is extracted from the full description. On the other scale the short-distance fluctuations are analyzed by the mean-field theory.

PACS numbers : 68.45.Gd, 68.35.Rh

## I. Introduction

The analysis of physical systems usually involves some reductions in the description of the state of the systems. This is also the case for inhomogeneous systems consisting of two coexisting bulk phases separated by an interface fluctuating in the presence of a substrate. Certain properties of such systems, e.g. those related to the adsorption phenomena, can be conveniently described with the help of a single mesoscopic variable which is the distance of the interface from the substrate.

In this paper we consider such a system. The substrate has the form of an infinite wedge extending along the  $y$ -direction with the opening angle  $2\varphi$ , see Fig.1. The quasi-bulk phase adsorbed on the substrate is denoted as the  $\beta$ -phase while the phase far above the substrate is denoted as the  $\alpha$ -phase. The shape of the substrate is given by  $z = |x| \cot \varphi$  and  $\ell(x, y)$  describes the distance of the  $\alpha$ - $\beta$  interface from this substrate. Recently, it was pointed out [1-7] that the above system may undergo the critical transition in which the position of the central part of the interface (above the edge of the wedge) moves to infinity while the asymptotic parts of the interface corresponding to  $|x| \rightarrow \infty$  remain close to the substrate. This interfacial transition is called the filling transition to distinguish it from the wetting transition taking place on planar substrates [8, 9]. Thermodynamically the filling transition point is located at the bulk  $\alpha$ - $\beta$  coexistence and the filling temperature (which depends on the wedge opening angle  $\varphi$ ) is denoted as  $T_\varphi$ ;  $T_\varphi < T_w$ , where  $T_w$  is the wetting temperature on the planar substrate.

In their recent paper Parry et al. [10] used the transfer-matrix method to evaluate - among others - the values of the critical indices associated with the interfacial behavior near the filling transition. For this purpose the next step in the reduction of the description was made. The two-dimensional interface  $\ell(x, y)$  was replaced by the one-dimensional mid-point line  $\ell(y) \equiv \ell(0, y)$  (see Fig.1) for which the appropriate Hamiltonian was proposed.

If, however, one is interested in the full two-dimensional structure of the fluctuating interface near the critical filling transition, i.e. also in the short-distance behavior which is not included in the reduced description then - at least in principle - one has to go beyond the mean-field analysis. In this paper we propose how to describe the two-dimensional interface close to the filling transition in the system with short-ranged forces [6]. We expect that the geometry-dependent effects are important for short distances. Since the mid-point height does not vary too much on the short length-scale the idea is to fix this height at some arbitrarily chosen point and to assume the mean-field profile of the interface in the vicinity of this chosen point (along the  $x$ -direction). Then one uses the mean-field approximation to describe the

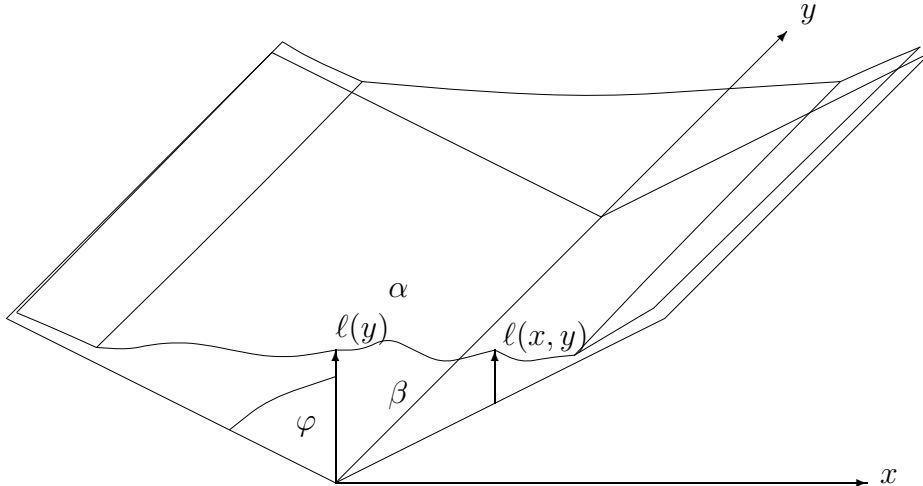


Figure 1: The wedge geometry and the fluctuating  $\alpha - \beta$  interface

”relative” fluctuations around the fixed point. The two-point height distribution function for neighboring points consists of two parts: the one-point distribution corresponding to one of the points (or the average height of them) and the conditional probability distribution in the form of a Gaussian with position-dependent dispersion. Such quantity does not diverge at the filling transition and may turn out useful when some geometry-dependent observables are considered. The mean-field description becomes then legitimate because by fixing the position of the interface and looking at the conditional distribution one forces the local fluctuations to be small and so one insists that the system is locally outside the critical region.

## II. The mean-field description

The interfacial Hamiltonian in the case of a very open wedge ( $\cot \varphi \ll 1$ ) has the standard form [6,10]

$$\begin{aligned}
 H[\ell] &= \int dx \int dy [\Sigma(\nabla(\ell + \alpha|x|)^2/2 + \omega(\ell) - \omega(\ell_\infty))] \\
 &= \int dx \int dy [\Sigma(\nabla\ell)^2/2 + \omega(\ell) - \omega(\ell_\infty)] - 2\Sigma\alpha \int dy[\ell(0, y) - \ell_\infty],
 \end{aligned} \tag{2.1}$$

where  $\ell(x, y)$  (Fig.1) denotes the width of the adsorbed  $\beta$ -like layer measured in the vertical direction and  $\Sigma$  is the  $\alpha$ - $\beta$  interfacial tension.  $\omega(\ell)$  denotes the interfacial pinning potential corresponding to the critical wetting in the pla-

nar case. For short-range forces considered in this paper it has the following form [6-11]

$$\omega(\ell) = -W t \exp(-\ell/\xi) + U \exp(-2\ell/\xi) , \quad (2.2)$$

where  $\xi$  is the bulk correlation length (in the  $\beta$ -phase),  $U$  and  $W$  are positive constants. (We use the convention in which the factor  $1/k_B T$  is included into the Hamiltonian.) The parameter  $t$  denotes the dimensionless deviation from the wetting temperature for the planar substrate, i.e.  $t > 0$  for  $T < T_w$  and  $t = 0$  for  $T = T_w$ .  $\ell_\infty$  is the equilibrium width of the adsorbed layer on the planar substrate which minimizes the potential  $\omega(\ell)$ :  $\exp(-\ell_\infty/\xi) = \frac{W}{2U}t$ . Because the wedge is very open we put  $\sin \varphi = 1$  and  $\cot \varphi = \cos \varphi = \alpha$ .

The mean-field profile  $\bar{\ell}(x)$  varies only in the  $x$  direction. It satisfies the Euler-Lagrange equation [6]

$$\Sigma \bar{\ell}''(x) = \omega'(\bar{\ell}) \quad (2.3)$$

and the boundary conditions:  $\bar{\ell}(\pm\infty) = \ell_\infty$ ,  $\bar{\ell}'(0_\pm) = \mp\alpha$ . The solution of Eq.(2.3) is

$$x(\bar{\ell}) = \pm \int_{\bar{\ell}}^{\ell_0} \frac{d\ell}{\sqrt{2(\omega(\ell) - \omega(\ell_\infty))/\Sigma}} , \quad (2.4)$$

where the width of the mean-field profile at the center of the wedge  $\ell_0 = \bar{\ell}(0)$  satisfies  $\omega(\ell_0) - \omega(\ell_\infty) = \Sigma\alpha^2/2$ . With the help of the Young equation one can relate  $\omega(\ell_\infty)$  to the contact angle  $\Theta$  on the planar substrate:  $-\omega(\ell_\infty) = \Sigma\Theta^2/2$ . From this we see that  $\omega(\ell_0) = \Sigma(\alpha^2 - \Theta^2)/2$  and the filling transition ( $\ell_0 \rightarrow \infty$ ,  $\ell_\infty$  - finite) takes place when  $\Theta(T = T_\varphi) = \alpha$ .

For small deviations  $\delta\ell(x, y) = \ell(x, y) - \bar{\ell}(x)$  from the mean field profile  $\bar{\ell}(x)$  the fluctuation Hamiltonian  $H_{fl}[\delta\ell] = H[\bar{\ell} + \delta\ell] - H[\bar{\ell}]$  is bilinear in  $\delta\ell$

$$H_{fl}[\delta\ell] = \int dx \int dy \frac{1}{2} [\Sigma(\nabla\delta\ell)^2 + \omega''(\bar{\ell})(\delta\ell)^2] . \quad (2.5)$$

The important feature of the critical filling transition is the existence of the *translational mode*, i.e. the fluctuation of the interface which requires very small energy (decreasing to 0 at the filling point). This fluctuation takes the form  $\delta\ell(x) = \epsilon|\bar{\ell}'(x)|$  and the corresponding energetical cost is  $H_{fl}[\delta\ell] = \epsilon^2\alpha\omega'(\ell_0)$ ; it decreases to 0 for  $\ell_0 \rightarrow \infty$ .

The corresponding differential equation for correlation function  $G(\mathbf{r}, \mathbf{r}') = \langle \delta\ell(\mathbf{r})\delta\ell(\mathbf{r}') \rangle$  has in the mean-field approximation the following form

$$[-\Sigma\Delta_{\mathbf{r}} + \omega''(\bar{\ell})] G(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') . \quad (2.6)$$

However, the mean-field description fails in case of the critical filling transition for short-ranged forces because Eq.(2.6) implies strong anisotropy of fluctuations of the interface. The fluctuations along the wedge diverge much faster than across the wedge. The latter are bounded by the geometry of the substrate. As shown in [10], the mean-field predictions are valid only for power-law forces of the type  $\omega(\ell) \sim \ell^{-p}$  for  $p < 4$ .

### III. The reduction of the order parameter

The *effective* way to analyze the critical fluctuations of  $\ell(x, y)$  near the filling transition point is to reduce the interfacial description by looking only at the mid-point height  $\ell(y) = \ell(0, y)$  [10]. In order to derive the corresponding Hamiltonian we proceed as follows: we minimize the Hamiltonian in Eq.(2.1) similarly as in the mean-field method but now with the constraint  $\ell(0, y) = \ell(y)$  imposed independently at each  $y$  [12]. From the corresponding Euler-Lagrange equation one obtains

$$x(\ell, y) = \pm \int_{\ell}^{\ell(y)} \frac{d\ell_1}{\sqrt{2(\omega(\ell_1) - \omega(\ell_\infty))/\Sigma}}. \quad (3.1)$$

As the result the one-dimensional Hamiltonian  $H_1[\ell(y)] = H[\ell(x, y)]$  valid for configurations given in Eq.(3.1) takes the form

$$\begin{aligned} H_1[\ell(y)] = \int dy \left\{ \frac{\Sigma(\ell'(y))^2 \int_{\ell_\infty}^{\ell(y)} d\ell_1 \sqrt{\Sigma(\omega(\ell_1) - \omega(\ell_\infty))/2}}{\omega(\ell(y)) - \omega(\ell_\infty)} \right. \\ \left. + 2 \int_{\ell_\infty}^{\ell(y)} d\ell_1 \left[ \sqrt{2\Sigma(\omega(\ell_1) - \omega(\ell_\infty))} - \alpha\Sigma \right] \right\}. \end{aligned} \quad (3.2)$$

For short-range forces, see Eq.(2.2), the above Hamiltonian can be explicitly evaluated

$$\begin{aligned} H_1[\ell] = \int dy \left\{ \frac{\Sigma(\tilde{\ell}'(y))^2}{\Theta} \frac{\tilde{\ell} - \xi(1 - \exp(-\tilde{\ell}/\xi))}{[1 - \exp(-\tilde{\ell}/\xi)]^2} \right. \\ \left. + 2\Sigma[(\Theta - \alpha)\tilde{\ell} - \Theta\xi(1 - \exp(-\tilde{\ell}/\xi))] \right\}, \end{aligned} \quad (3.3)$$

where  $\tilde{\ell}(y) = \ell(y) - \ell_\infty$ . For temperatures close to the filling transition one has  $\ell \gg \xi$  and  $\ell \gg \ell_\infty$  and Eq.(3.3) reduces to the one-dimensional Hamiltonian proposed phenomenologically in [10]

$$H_1[\ell(y)] \approx \int dy \left[ \frac{\Sigma\ell(y)}{\Theta} (\ell'(y))^2 + 2\Sigma(\Theta - \alpha)\ell(y) \right]. \quad (3.4)$$

The above Hamiltonian has relatively simple structure and is easy to renormalize. After introducing the rescaled variables  $L$  and  $Y$

$$\Theta y = (2\Sigma)^{-1/2}(\Theta/\alpha - 1)^{-3/4}Y, \quad \ell = (2\Sigma)^{-1/2}(\Theta/\alpha - 1)^{-1/4}L \quad (3.5)$$

it takes the form

$$H_1[L(Y)] = \int dY \left[ \frac{L(Y)}{2} (L'(Y))^2 + L(Y) \right], \quad (3.6)$$

which is free from parameters. Accordingly, the critical behavior of the mean mid-point height  $\langle \ell(y) \rangle$  and the correlation length  $\xi_y$  follow directly from the above rescaling:  $\langle \ell(y) \rangle \sim (\Theta - \alpha)^{-1/4}$  and  $\xi_y \sim (\Theta - \alpha)^{-3/4}$ . The values of the critical indices agree with those obtained in [10].

The one-dimensional model described by the Hamiltonian in Eq.(3.6) can be solved via the transfer-matrix method [10, 14]. However, in this method the presence of the factor  $L(Y)$  in front of  $(L'(Y))^2$  is the source of ambiguity while discretizing the problem and defining the measure which is then used to evaluate the relevant propagator [13]. In order to avoid such problems it is convenient to introduce the new variable  $\eta \equiv 2L^{3/2}/3$  which "absorbs" the dangerous factor  $L(Y)$  in front of  $(L'(Y))^2$ . Then the Hamiltonian takes the form

$$H_1[\eta(Y)] = \int dY \left[ (\eta'(Y))^2/2 + (3\eta/2)^{2/3} \right]. \quad (3.7)$$

The corresponding propagator

$$V(\eta_2, \eta_1, Y) = \int \mathcal{D}\eta \exp(-H_1[\eta]) \Big|_{\eta(0)=\eta_1}^{\eta(Y)=\eta_2} \quad (3.8)$$

can be evaluated by solving - within the transfer matrix approach [14] - the following equation

$$\frac{\partial V}{\partial Y} = \frac{\partial^2 V}{2\partial \eta_2^2} - (2\eta_2/3)^{2/3}V. \quad (3.9)$$

This equation must be supplemented by the appropriate boundary condition for  $\eta_2 = 0$ . The general form of such a condition

$$\partial_{\eta_2} \ln V|_{\eta_2=0} = a_t$$

is similar to that found in [14] for 2D wetting. In the present case the parameter  $\eta$  is  $t$ -dependent so one expects the  $t$ -independent boundary condition

$$\partial_{\ell_2^{3/2}} \ln V = a = (2\Sigma)^{3/4}(\Theta/\alpha - 1)^{3/8}a_t.$$

For  $a < 0$  the edge effects become dominant and no filling is observed. Thus we assume  $a > 0$  ( $a^{-2/3}$  is the range of the influence of the edge effects), so  $a_t \sim (\Theta - \alpha)^{-3/8}$  in the critical region. The appropriate boundary condition is then  $V(0, \eta_1, Y) = 0$ .

The propagator  $V(\eta_2, \eta_1, Y)$  can be expressed by normalized eigenfunctions  $\psi_n(\eta)$  and eigenvalues  $E_n$  of the equation

$$E_n \psi_n = -\frac{\partial^2 \psi_n}{2\partial \eta^2} + (3\eta/2)^{2/3} \psi_n. \quad (3.10)$$

Then

$$V(\eta_2, \eta_1, Y) = \sum_n \psi_n(\eta_1) \psi_n(\eta_2) e^{-E_n Y}. \quad (3.11)$$

The probability distribution of the mid-point height is given by  $\psi_0^2(\eta)$  and other quantities can be expressed by the appropriate combinations of eigenfunctions.

#### IV. The short-distance correlation function

Obviously the above one-dimensional approximation cannot describe the full two-dimensional structure of the interface. However, there are two different length-scales in this problem. The one-dimensional character of the filling transition is seen on scales  $\alpha y \sim \Sigma \ell^3$  while the two-dimensional structure becomes important when  $\alpha y \sim \ell$ . In the critical region these two scales are well separated.

Therefore, in order to analyze the short-distance behavior one can introduce *the conditional correlation function*. This is done in the following way. We assume that for certain  $y_0$  (for convenience we set  $y_0 = 0$ ) the interface profile  $\ell(x, y)$  is constrained:  $\ell(x, y_0 = 0) = \bar{\ell}(x)$ , where  $\bar{\ell}(x)$  is described by Eq.(2.4) but with *given*  $\ell_0$ . The full Hamiltonian is then expanded in the Taylor series in the variable  $\phi(x, y) = \ell(x, y) - \ell(x, 0)$  up to  $\phi^2$  terms. In this way one obtains (up to the constant term)

$$\begin{aligned} H[\phi] = & -2 \int dy \left[ \alpha \Sigma - \sqrt{2\Sigma(\omega(\ell_0) - \omega(\ell_\infty))} \right] \phi(0, y) \\ & + \frac{1}{2} \int dx \int dy \left[ \Sigma(\nabla \phi)^2 + \omega''(\bar{\ell}(x)) \phi^2 \right]. \end{aligned} \quad (4.1)$$

The first term on the rhs of Eq.(4.1) is very small in the critical region (i.e. for  $\ell_0 \rightarrow \infty$  and  $\Theta \approx \alpha$ ); it is given by  $\sim 2\Sigma(\Theta - \alpha)$ . Thus for short distances one keeps only the second term. The resulting structure of the

Hamiltonian implies the following differential equation for the conditional correlation function

$$G_{\ell_0}(\mathbf{r}, \mathbf{r}') = \langle \phi(\mathbf{r}) \phi(\mathbf{r}') \rangle|_{\ell(x,0)=\bar{\ell}(x)}$$

$$[-\Sigma\Delta_{\mathbf{r}} + \omega''(\bar{\ell})] G_{\ell_0}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}') . \quad (4.2)$$

Similarly as in Eq.(3.11) the conditional correlation function can be expressed by the normalized eigenfunctions  $\psi_{\mathbf{q}}$  and eigenvalues  $E_{\mathbf{q}}$  of the operator  $[-\Sigma\Delta + \omega''(\bar{\ell})]$

$$G_{\ell_0}(\mathbf{r}, \mathbf{r}') = \sum_{\mathbf{q}} \frac{\psi_{\mathbf{q}}(\mathbf{r}) \psi_{\mathbf{q}}(\mathbf{r}')}{E_{\mathbf{q}}} .$$

In this approach one has to analyze carefully the contribution from the eigenvalues tending to 0. One expects that the eigenfunctions with the lowest eigenvalues will have their structure similar to  $\psi_0 = |\bar{\ell}'(x)|$  which itself corresponds to the translational mode (although it does not satisfy the appropriate boundary condition in the present case). Thus we introduce the new variables  $\psi_{\mathbf{q}} = \varphi_{\mathbf{q}} \psi_0$  and the equation for  $\varphi_{\mathbf{q}}$  has the form

$$[E_{\mathbf{q}} + \Sigma\Delta] \varphi_{\mathbf{q}} = -2 \left[ \sqrt{2\Sigma(\omega(\bar{\ell}) - \omega(\ell_{\infty}))} \right]' \partial_x \varphi_{\mathbf{q}} . \quad (4.3)$$

The expression on the rhs of the above equation is close to 0 for  $\alpha|x| < \ell_0$  and for  $\alpha|x|$  approaching  $\ell_0$  it quickly becomes equal to  $-2\sqrt{\Sigma\omega''(\ell_{\infty})}\partial_x \varphi_{\mathbf{q}}$ . We are interested only in the long-wave fluctuations such that  $E_{\mathbf{q}} \sim \Sigma(\alpha/\ell_0)^2$ . If all terms in the above equation are to be of the same order of magnitude for  $\alpha|x| > \ell_0$  then one should have  $\partial_x \ln \varphi_{\mathbf{q}} \sim E_{\mathbf{q}} \xi_{\pi} / \Sigma \sim \xi_{\pi} \alpha^2 / \ell_0^2$ , where  $\xi_{\pi} = (\omega''(\ell_{\infty})/\Sigma)^{-1/2}$  is the correlation length for the planar case. Note that  $\xi_{\pi}$  which diverges at the critical wetting on the planar substrate remains finite at the critical filling transition.

The above considerations lead to the following equation for the conditional correlation function for  $\alpha|x| < \ell_0$ , i.e for the central "free" part of the interface

$$-\Sigma\Delta_{\mathbf{r}} G_{\ell_0}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r}_2 - \mathbf{r}_1) \quad (4.4)$$

$$\partial_x G_{\ell_0}(\mathbf{r}, \mathbf{r}')|_{|x|=\ell_0/\alpha} = 0. \quad (4.5)$$

One also needs the boundary condition for  $y' \rightarrow \infty$ . Since there exists the long-range order on the scale considered now, i.e. for  $\alpha y \sim \ell_0$  one should not



expect  $G_{\ell_0}(\mathbf{r}_1, \mathbf{r}_2) \xrightarrow{\mathbf{r}_2 \rightarrow \infty} 0$ . Instead we assume  $G_{\ell_0}(\mathbf{r}_1, \mathbf{r}_2) \xrightarrow{\mathbf{r}_2 \rightarrow \infty} f(\mathbf{r}_1) < \infty$ , i.e.  $G$  remains finite. Using the standard methods of conformal transformations (see Appendix) one obtains the following solution of Eqs.(4.4, 4.5)

$$\begin{aligned}
G_{\ell_0}(\mathbf{r}_1, \mathbf{r}_2) = & -\frac{1}{4\Sigma\pi} \left[ \ln \left( e^{(Y_1-Y_2)\pi/2} + e^{(Y_2-Y_1)\pi/2} - 2 \cos(X_1 - X_2)\pi/2 \right) \right. \\
& + \ln \left( e^{(Y_1-Y_2)\pi/2} + e^{(Y_2-Y_1)\pi/2} + 2 \cos(X_1 + X_2)\pi/2 \right) \\
& - \ln \left( e^{(Y_1+Y_2)\pi} + 1 - 2e^{(Y_1+Y_2)\pi/2} \cos(X_1 - X_2)\pi/2 \right) \\
& \left. - \ln \left( e^{(Y_1+Y_2)\pi} + 1 + 2e^{(Y_1+Y_2)\pi/2} \cos(X_1 + X_2)\pi/2 \right) + \pi(Y_1 + Y_2) \right]
\end{aligned} \tag{4.6}$$

where  $\mathbf{R}_i = \alpha \mathbf{r}_i / \ell_0$ ,  $i = 1, 2$ . We note that for  $\mathbf{r}_2 \rightarrow \infty$  one has  $G_{\ell_0}(\mathbf{r}_1, \mathbf{r}_2) \rightarrow \alpha y_1 / \Sigma \ell_0$ .

## V. The short-distance dispersion

For short distances the two-point  $\ell$ -distribution function has the form

$$p(\ell_1, \mathbf{r}_1; \ell_2, \mathbf{r}_2) \approx p(\ell_0) \frac{\exp \left( -(\ell_2 - \ell_1)^2 / 2\sigma(\mathbf{r}_1, \mathbf{r}_2, \ell_0) \right)}{[2\pi\sigma(\mathbf{r}_1, \mathbf{r}_2, \ell_0)]^{1/2}} \tag{5.1}$$

where  $\ell_0$  is the height of the interface above the edge of the wedge

$$\ell_0 = (\ell_1 + \ell_2 + \alpha(|x_1| + |x_2|))/2 \approx \ell_1 + \alpha|x_1| \approx \ell_2 + \alpha|x_2|.$$

We use the conditional correlation function  $G_{\ell_0}$  to obtain the expression for the dispersion  $\sigma$

$$\sigma = \langle (\ell_2 - \ell_1)^2 \rangle = G_{\ell_0}(\mathbf{r}_1, \mathbf{r}_1) - 2G_{\ell_0}(\mathbf{r}_1, \mathbf{r}_2) + G_{\ell_0}(\mathbf{r}_2, \mathbf{r}_2). \tag{5.2}$$

The standard problem which one encounters at this point is that  $G_{\ell_0}(\mathbf{r}_1, \mathbf{r}_2)$  diverges for  $\mathbf{r}_2 \rightarrow \mathbf{r}_1$  [15, 16]. This divergence can be removed by regularizing the function  $G_{\ell_0}(\mathbf{r}_1, \mathbf{r}_2)$ , e.g. by adding to the Hamiltonian given in Eq.(4.1) the term  $a^2(\Delta\phi)^2/2$ , where  $a$  is a dimensionless parameter. This procedure yields the following equation for the regularized function  $G_{\ell_0}^{(a)}(\mathbf{r}_1, \mathbf{r}_2)$

$$[a^2\Delta_{\mathbf{r}_1}^2 - \Sigma\Delta_{\mathbf{r}_1}] G_{\ell_0}^{(a)} = \delta(\mathbf{r}_2 - \mathbf{r}_1).$$

For small  $a$  the solution of the above equation has the form

$$G_{\ell_0}^{(a)}(\mathbf{r}_1, \mathbf{r}_2) = G_{\ell_0}(\mathbf{r}_1, \mathbf{r}_2) - K_0(\Sigma^{1/2}|\mathbf{r}_2 - \mathbf{r}_1|/a)/2\pi\Sigma,$$

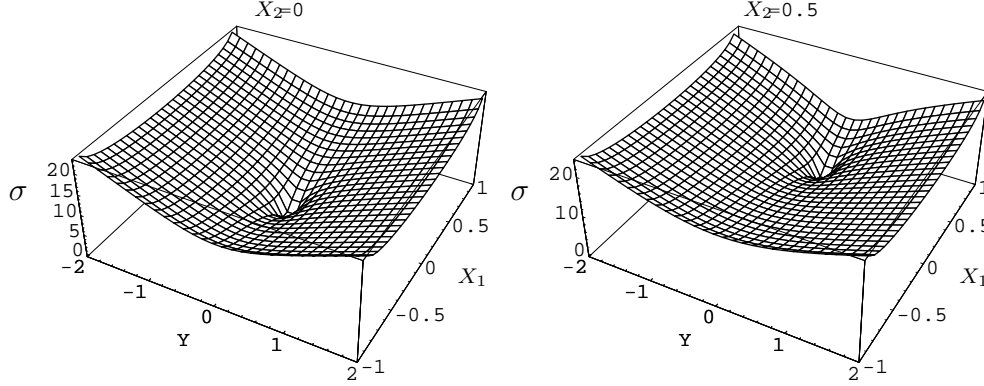


Figure 2: The dimensionless dispersion  $\sigma$  as function of  $X_1$  and  $Y$  for  $X_2 = 0$  and  $0.5$ , respectively.

where  $K_0$  is the modified Bessel function. In this way the short-distance divergence is removed and one has

$$G_{\ell_0}^{(a)}(\mathbf{r}_1, \mathbf{r}_1) = \lim_{\mathbf{r}_2 \rightarrow \mathbf{r}_1} \left[ G_{\ell_0}(\mathbf{r}_1, \mathbf{r}_2) + \frac{\gamma + \ln(\Sigma^{1/2}|\mathbf{r}_2 - \mathbf{r}_1|/2a)}{2\pi\Sigma} \right],$$

where  $\gamma$  is the Euler constant. Now the expression for the dispersion  $\sigma(\mathbf{r}_1, \mathbf{r}_2, \ell_0)$ , Eq.(5.2) can be written down explicitly. We are interested in the situation in which the constraint affects only the mean height of the interface and thus we consider the case  $y_1, y_2 \gg \ell_0/\alpha$  and  $|\mathbf{r}_1 - \mathbf{r}_2| \gg a$ . Then

$$\begin{aligned} \sigma(\mathbf{r}_1, \mathbf{r}_2, \ell_0) = & \frac{1}{2\Sigma\pi} \{ 2 \ln(\ell_0 \Sigma^{1/2}/a\pi\alpha) + 2\gamma - \ln(\cos(X_1\pi/2)) \} \\ & - \ln(\cos(X_2\pi/2)) + \ln[\text{ch}((Y_1 - Y_2)\pi/2) - \cos((X_1 - X_2)\pi/2)] \\ & + \ln[\text{ch}((Y_1 + Y_2)\pi/2) + \cos((X_1 + X_2)\pi/2)] \}. \end{aligned} \quad (5.3)$$

The behavior of  $\sigma$  is shown on Fig.2. We see that in this limit  $\sigma(\mathbf{r}_1, \mathbf{r}_2, \ell_0)$  depends - in addition to  $X_1$  and  $X_2$  - only on the distance  $Y = Y_2 - Y_1$ . For fixed values of  $X_1$  and  $X_2$  it is an increasing function of  $|Y|$ , see Fig.2. Thus the relative fluctuations of the interface position at points distant along the edge of the wedge become large.

It is interesting to observe that for  $|y_1 - y_2| \gg \ell_0/\alpha$  one gets

$$\sigma(\mathbf{r}_1, \mathbf{r}_2, \ell_0) \approx |y_1 - y_2| \alpha (\Sigma \ell_0)^{-1} / 2.$$

This result agrees with the prediction of the one-dimensional model valid on the scale where  $\Sigma \ell_0^3 \gg \alpha |y_1 - y_2|$ . It can be derived with the help of Eq.(3.9).

Thus the results obtained via the conditional correlation function in Chapters IV and V are consistent with those stemming from the transfer-matrix analysis of the 1D model in Chapter III.

## VI. Conclusions

The reduced description of the interface fluctuating in the presence of the wedge-shaped substrate is derived in an explicit way. This reduced description is based on the one-dimensional Hamiltonian [10] and the presented derivation of this Hamiltonian makes clear use of the physical assumptions behind it. Although the one-dimensional Hamiltonian allows one to find the relevant critical exponents it cannot describe the full two-dimensional structure of the interface. We have proposed the method of supplementing this one-dimensional picture by the local two-dimensional constrained fluctuations which can be analyzed within the mean-field theory and described by the conditional correlation function. These fluctuations are not divergent at the filling transition. The proposed method can be used to calculate the geometry-dependent observables. Moreover, it predicts the behavior of the dispersion of the conditional correlation function which agrees with the predictions of the one-dimensional model.

**Acknowledgment** The authors gratefully acknowledge the discussions with H.W.Diehl, S.Dietrich, and A.Parry, and the support by the Foundation for German-Polish Collaboration under Grant. No. 3269/97/LN.

## Appendix

In this Appendix we sketch the consecutive steps of the method of conformal transformations which lead to the solution of Eqs.(4.4,4.5).

After introducing the complex variables  $\mathbf{r}_{1,2} = (x_{1,2}, y_{1,2})$ ,  $z_{1,2} = x_{1,2} + iy_{1,2}$  Eq.(4.4) can be rewritten as

$$-4\Sigma\partial_{z_1}\partial_{\bar{z}_1}G_{\ell_0}(z_1, z_2) = \delta(z_1 - z_2). \quad (\text{A.1})$$

together with the boundary condition (Eq.(4.5))

$$i[dz_c\partial_z - d\bar{z}_c\partial_{\bar{z}}]G_{\ell_0}(z, z_2) = 0, \quad (\text{A.2})$$

where  $z_c$  denotes the contour on which the boundary condition is given. Eqs.(A.1, A.2) are invariant with respect to the conformal transformations. The solution of Eq.(A.1) valid for the whole plane has the form

$$G_{\infty}(z_1, z_2) = -\frac{1}{4\Sigma\pi}[\log(z_1 - z_2) + \log(\bar{z}_1 - \bar{z}_2)]. \quad (\text{A.3})$$

The solution valid for the semi-plane  $\Re z > 0$  with the Neumann condition on  $\Re z = 0$  is found with the help of the method of images and has the form

$$G_{\Re z > 0}(z_1, z_2) = G_{\infty}(z_1, z_2) + G_{\infty}(z_1, -\bar{z}_2). \quad (\text{A.4})$$

After introducing the conformal transformation  $Z \mapsto e^{-i\pi Z/2}$  for dimensionless variables  $Z = z\alpha/\ell$ ,  $\mathbf{R} = \mathbf{r}\alpha/\ell$  one obtains

$$G_{(\ell)}(Z_1, Z_2) = G_{\Re Z > 0}(e^{i\pi Z_1/2}, e^{i\pi Z_2/2}) \quad (\text{A.5})$$

with the Neumann condition at  $X = \pm 1$ . Finally, taking into account the constraint imposed on the interface at  $y = 0$  and using the freedom to add to the rhs of Eq.(A.4) the solutions of the Laplace equation leads to the solution of Eq.(4.4) given in Eq.(4.6).

## References

- [1] Y. Pomeau, J. Coll. Interf. Sci. **113**, 5 (1985).
- [2] E. Cheng and M.W. Cole, Phys. Rev. B **41**, 9650 (1990).
- [3] P.M. Duxbury and A.C. Orrick, Phys. Rev. B **39**, 2944 (1989).
- [4] E.H. Hauge, Phys. Rev. A **46**, 4944 (1992).
- [5] M. Napiórkowski, W. Koch, and S. Dietrich, Phys. Rev. A **45**, 5760 (1992).
- [6] K. Rejmer, S. Dietrich and M. Napiórkowski, *Phys. Rev.* **E60**, 4027 (1999), `\protect\vrule width0pt\protect\href{http://arXiv.org/abs/cond-mat/9812115}`
- [7] S. Dietrich, in *New Approaches to Old and New Problems in Liquid State Theory - Inhomogeneities and Phase Separation in Simple, Complex and Quantum Fluids*, Vol.**C529** of *NATO Advanced Study Institute, Messina, Italy, 1998*, edited by C. Caccamo, J.P.Hansen and G.Stell (Kluwer, Dordrecht,1999), p.197.
- [8] S. Dietrich, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J.L. Lebowitz (Academic, London, 1988), Vol.12, p. 1.
- [9] M. Schick, in *Liquids at Interfaces, Proceedings of the Les Houches Summer School in Theoretical Physics, Session XLVIII*, edited by J. Chavrolin, J. F. Joanny, and J. Zinn-Justin (North-Holland, Amsterdam, 1990), p. 415.
- [10] A. O. Parry, C. Rascon and A. J. Wood, `\protect\vrule width0pt\protect\href{http://arXiv.org/abs/cond-mat/9911431}`{cond
- [11] R. Lipowsky, D. M. Kroll and R. K. P. Zia, *Phys. Rev.* **B27**, 4499 (1983).
- [12] M.E.Fisher, A.J.Jin, and A.O.Parry, Ber. der Bunsengesellschaft, **68**, 357 (1994).
- [13] A.Bednorz and M.Napiórkowski, in preparation.
- [14] T. Burkhardt, *Phys. Rev.* **B40**, 6987 (1989).
- [15] H.W. Diehl, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and J.L. Lebowitz (Academic, London, 1986), Vol.10, p. 75.

- [16] F.Brezin, J.C.Guillou, and J.Zinn-Justin, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M.S.Green (Academic, London, 1976), Vol.6, p. 127.